

Supersymmetric Quantum Mechanics and the Gauss-Bonnet-Chern Theorem

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Abstract

Following Feynman's prescription for constructing a path integral representation of the propagator of a quantum theory, a short-time approximation to the propagator for imaginary time, $N = 1$ supersymmetric quantum mechanics on a compact, even-dimensional Riemannian manifold is constructed. The path integral is interpreted as the limit of products of the approximate short-time propagator over a partition of a finite time interval. This limit is shown to converge uniformly for finite time as kernels, and to admit a version of steepest descent approximation sufficiently robust that the path integral proof of the Gauss-Bonnet-Chern Theorem becomes directly rigorous.

Introduction

Richard Feynman [Fey42], building on ideas of Dirac [Dir33], identified the short-time approximation of the kernel of the propagator for quantum mechanics and, by extension, any quantum theory with the product of the exponential of i/\hbar times the classical action and a certain coefficient. The full propagator should be the limit of the product of many short-time approximations, which can be readily reformulated (sweeping the infinite product of the aforementioned coefficient into the measure) as an integral over all possible paths (or, in general, field configurations) of the exponential of i/\hbar times the action of the path. This is the path integral formulation of a quantum theory. Feynman's path integral has proven an extremely powerful tool in quantum theory, and, surprisingly, has proven a powerful heuristic tool in mathematics as well. Adept use of path integral reasoning has constructed elegant arguments for ostensibly purely mathematical propositions whose proofs otherwise were unknown or required deep and quite different arguments [Wit89, Wit91, SW94, Wit94]. One of the simplest is the path integral proofs of various versions of the Atiyah-Singer index theorem, including for twisted versions of the Dirac and DeRham complexes (the latter giving both the Gauss-Bonnet-Chern theorem and the Hirzebruch signature theorem).

These heuristic "proofs" are due independently to Alvarez-Gaumé [AG83] and Friedan and Windey [FW84] based on an approach suggested by Witten [Wit82a, Wit82b]. The argument (focusing exclusively on the GBC case for simplicity) begins with a heuristic identifying on the one hand the path integral for imaginary-time $N = 1$ supersymmetric quantum mechanics (SUSYQM) taken over loops with anti-periodic boundary conditions for the fermions with on the other hand the supertrace of the kernel of the propagator, i.e. of the heat kernel. An argument of McKean and Singer [MS67] shows the supertrace is equal to the signed sum of the Betti numbers, independent of the time t . On the other hand,

stationary phase, which in the imaginary time case becomes steepest descent, applied formally to this path integral equates the small- t limit of the integral over loops to the Pfaffian of curvature, recovering the GBC Theorem.

This approach to GBC represents arguably the simplest use of path integral reasoning to arrive at a clearly nontrivial purely mathematical conclusion. As such it is an important benchmark for a rigorous interpretation of the path integral. A rigorous interpretation of a heuristic reasoning tool should offer a reasonably direct way to translate heuristic arguments into rigorous proofs. In particular, a construction of the path integral for the supersymmetric propagator that is robust enough to turn the path integral argument of the GBC Theorem into a real proof would represent an incremental step towards understanding similar arguments rigorously. This is the goal of the present paper.

In fact, there is closely-related work in the literature. Bismut [Bis84a] uses stochastic techniques with the heat equation to give a proof of the index theorem in the spirit of the physics argument. Rogers [Rog87] also uses stochastic techniques to construct an explicit supersymmetric path integral and reproduce the path integral proof of GBC (and, in [Rog92b], the twisted Hirzebruch index theorem, from which follows the full index theorem). Andersson and Driver [AD99], use stochastic techniques to construct a version of the bosonic path integral on curved space, though not towards the end of proving an index theorem. The innovation in this paper is to make rigorous Feynman's original time-slicing procedure and to use the resulting path integral to prove the Gauss-Bonnet-Chern theorem.

Feynman was quite clear there were two ambiguities in his formulation; namely, the specification of the coefficient and the existence of the limit. The latter is a straightforward-enough question, and its answer will take up the bulk of the paper. The former is subtle and requires some discussion.

The path integral formalism suggests the propagator for a quantum theory (and by extension the quantum Hamiltonian) is completely determined by the action functional plus whatever goes into defining the putative measure on the space of world sheets, which one might hope would involve no more than kinematic aspects of the theory. If true this would be quite a remarkable claim, because purely classical data would determine the quantum Hamiltonian. In the canonical approach quantizing the classical Hamiltonian typically requires a choice of how to order products of position and momentum operators, and we do not expect the resolution of such operator ordering ambiguities to be determined by the classical theory for all observables. The hope would be that some universal procedure picks out, based on the classical data for a theory, the particular choice of operator ordering for the quantum Hamiltonian that nature prefers.

From the Feynman integral point of view this choice would appear in the coefficient discussed above. More specifically, Feynman proposed approximating the kernel of the propagator by

$$K(x, y; t) = C(x, y; t) e^{iS(x, y; t)/\hbar}$$

where x and y are position variables in the initial and final states and S is

the classical action. To have a hope of converging properly this approximate kernel must agree with the kernel of the propagator to first order in t , which means that its derivative at $t = 0$ should give the quantum Hamiltonian operator. The situation already becomes subtle in bosonic quantum mechanics on curved space. There the classical Hamiltonian is $p_i g^{ij}(x) p_j$, so the quantum Hamiltonian should be the Laplace-Beltrami operator plus terms resolving operator-ordering ambiguities. These would involve one or no spatial derivatives as well as traces of the curvature and higher order terms in the metric.

There are several ways to choose C and thus specify a short-time approximation to the propagator. Geometric quantization [Woo92] (specifically the BKS pairing) builds C out of what is essentially a Jacobian factor for the symplectic form [DeW57], yielding the Van Vleck determinant. Assuming the Laplace-Beltrami operator is the exact quantum Hamiltonian, the WKB approximation gives the same C (in the normalization of this paper, $C = e^{\text{Ricci}(\mathbf{y}_x, \mathbf{y}_x)/12}$ to the appropriate order). The presence of this term in the approximate kernel proves to be essential to get appropriate convergence to the kernel of a quantum Hamiltonian which is the Laplace-Beltrami operator plus $\mathfrak{r}/6$, where \mathfrak{r} is the scalar curvature. Various other regularizations of the path integral yield Laplace-Beltrami plus other multiples of the scalar curvature [Ful96]. Aesthetic considerations lead mathematicians to hope and expect that the Laplace-Beltrami operator is the “correct” Hamiltonian on the nose. There is no realistic hope of determining which of these nature prefers, as it seems unlikely the difference between these Hamiltonians could be accessible to any foreseeable experiment [Sch81].

In the presence of all this confusion, the authors have abandoned the effort to offer a coherent procedure for choosing C based only on the action and the kinematics of the theory, and will choose it based on the desired Hamiltonian (which in the case at hand is the Laplace-de Rham operator on forms). That is, we choose C so that $C e^{-S(x,y;t)}$ is a solution to the Laplace-de Rham heat equation to zeroth order in t (and has total integral 1 to zeroth order). Inspection of the resulting approximate kernel in Eq. (3.1) shows the Ricci curvature term corresponding to the Van Vleck term in the bosonic case, and a scalar curvature term which resolves operator-ordering ambiguities coming from both the bosonic piece and the four fermi term.

One final technical point is in order. In general the metaplectic correction suggests replacing the naive notion of the Hilbert space in quantum theory, which is to say functions on position space, with half-densities or half-forms on this space [Woo92]. In the bosonic case this appears in the Van Vleck determinant, which is naturally viewed as a half-density on the product space. For supersymmetric quantum mechanics the naive Hilbert space is forms on the manifold, which corresponds (see Section 2) to the space of functions of a Grassman variable. Because bosonic and Grassmannian determinants cancel, half densities on this space can be naturally identified with the naive space of functions. The upshot of this observation is that the remainder of the paper will ignore the metaplectic correction and half-densities.

TECHNICAL INTRODUCTION

The principal result of this paper is Theorem 6.1 which shows that the large partition limit of the kernel K defined in Eq. (3.1) converges to the heat kernel. Arguably the large partition limit captures the essence of the Feynman path integral while K captures the small- t approximation of the integrand in the path integral representation of the SUSY QM propagator for imaginary time, so the theorem may be viewed as a rigorous construction of the path integral for the imaginary-time propagator. Moreover, Eq. (6.2) and its special case Eq. (6.3), which relate the large-partition limit explicitly to the discretized path integral, provide a precise statement of the steepest-descent approximation to the path integral. The Gauss-Bonnet-Chern Theorem (Thm. 7.1) follows directly from Eq. (6.3), but only because the error term in this equation, expressing the difference between the path integral and its steepest-descent approximation, is bounded in a norm calibrated to send the supertrace of this error term to zero.

The norm is somewhat complicated because it serves two masters: On the one hand, the error terms in the large partition limit of K must be bounded in this norm, which means it must be designed so that, with errors measured in this norm, K almost satisfies the heat equation, the norm is almost multiplicative under $*$, and K is almost a contraction, as expressed in Eqs. (4.8), (4.7), and (4.6), respectively. On the other hand, as above, it must ensure that the supertrace of the error terms in Eq. (6.3) must go to zero in order for Theorem 7.1 to hold. More explicitly, Eq. (4.8) requires a bound on error terms of the form of polynomials times Gaussians, which become more sharply peaked as t approaches 0. The norm compares these terms near the diagonal to Gaussians and away from the diagonal to powers of t (in fact t^1 suffices) times a constant. This would completely suffice were it not for the second consideration; in fact, Eqs. (4.6) and (4.7) could be strengthened and simplified. However, to get the behavior under the supertrace that leads to Gauss-Bonnet-Chern not only must the error terms be bounded by t to a power greater than 1, but the norm must take into account that these errors are forms; ultimately, the top degree piece of the error must be bounded by a power of t greater than $n/2$. This would suggest choosing the norm so that a multiform of highest degree k and constant in t has norm $t^{-1/2}$ to the power $\max(0, k - 2)$. Such a norm still works for Eqs. (4.8) and (4.7), but, unfortunately, just barely fails for Eq. (4.6). So amend the norm again and instead of $t^{-1/2}$ use $t^{-1/2+\epsilon/n}$ for a sufficiently small ϵ . This somewhat unintuitive choice satisfies all the conditions.

The structure of the paper is as follows. Section 1 reviews standard results from differential geometry and Riemann normal coordinates, in particular describing approximate solutions to generalized heat equations. This will all be used towards the proof of Eq. (4.8). Section 2 reviews standard facts and notation of Grassman variables, and defines the norm used in the proofs of Eq. (4.8) and of Eqs. (4.6) and (4.7) (as well as of Theorem 7.1). Section 3 proves Eq. (3.5), from which Eq. (4.8) follows easily. Sections 4 and 5 prove Eqs. (4.6) and (4.7), as well as Eq. (4.9), which follows from Eq. (3.5). Sections 6 and 7 use only Eqs. (4.6), (4.7), and (4.9) to prove Theorem 6.1 showing that

the large partition limit converges to the propagator and providing bounds on the error. Finally Section 8 proves the Gauss-Bonnet Chern Theorem, using Theorem 6.1 and the property of the norm with regard to the supertrace.

1 Riemann Normal Coordinates and Kernels

The following material on Riemann normal coordinates is standard and can be found for example in [Ber], and follows straightforwardly from [BGV04].

Lemma 1.1 is used only in the proof of Proposition 4.1 and is elementary. Lemma 1.2 gives approximate fundamental solutions for some generalized heat equations in Riemann normal coordinates; it is used in the proof of Proposition 3.1 and is purely computational.

1.1 RIEMANN NORMAL COORDINATES

Let M be a fixed compact oriented Riemannian manifold of even dimension n . Since M is compact there is an injectivity radius such that for all $y \in M$ the exponential map $\exp_y: T_y M \rightarrow M$ is invertible on points within the injectivity radius of y (with respect to the Riemann distance). Refer to the inverse image of $x \in M$ within the injectivity radius as $\mathbf{x}_y \in T_y M$ (if the subscript is obvious omit it). A choice of orthonormal coordinates $(\partial_i)_{i=1}^n$ on $T_y M$ then gives coordinates on a patch around M which are called Riemann normal coordinates. In such coordinates express the metric $g_{ij}(\mathbf{x}) = (\partial_i, \partial_j)_x$, its inverse $g^{ij}(\mathbf{x})$, the Levi Civita connection $\Gamma_{ij}^k(\mathbf{x}) = g^{kl}(\partial_l, \nabla_{\partial_i} \partial_j)_x$ and the Levi Civita parallel transport from y to x along the unique geodesic connecting them $\mathfrak{P}_{ij}(\mathbf{x}) = (\partial_i, \mathfrak{P}_x^y \partial_j)_x$, in terms of the curvature at the origin y by the following formulas

$$g_{ij}(\mathbf{x}) = \delta_{ij} + \frac{1}{3} R_{ikjl}(0) x^k x^l + \mathcal{O}(|\mathbf{x}|^3) \quad (1.1)$$

$$g^{ij}(\mathbf{x}) = \delta^{ij} - \frac{1}{3} \delta^{ii'} R_{ki'l}^j(0) x^k x^l + \mathcal{O}(|\mathbf{x}|^3) \quad (1.2)$$

$$\Gamma_{ij}^k(\mathbf{x}) = -\frac{1}{3} [R_{ilj}^k(0) + R_{jli}^k(0)] x^l + \mathcal{O}(|\mathbf{x}|^2) \quad (1.3)$$

$$\mathfrak{P}_{ij}(\mathbf{x}) = \delta_{ij} + \frac{1}{6} R_{ikjl}(0) x^k x^l + \mathcal{O}(|\mathbf{x}|^3) \quad (1.4)$$

where $\mathcal{O}(|\mathbf{x}|^3)$ refers to an unnamed function which is bounded by some constant multiple of $|\mathbf{x}|^3$. Here the components of the Riemann curvature at x satisfy $R[\partial_i, \partial_j] \partial_k = R_{ijk}^l \partial_l(\mathbf{x})$ and $R_{ijkl}(\mathbf{x}) = (\partial_l, R[\partial_i, \partial_j] \partial_k)_x$.

The Ricci and scalar curvatures are

$$\text{Ricci}_{ij}(\mathbf{x}) = g^{kl}(\mathbf{x}) R_{kijl}(\mathbf{x}) \quad (1.5)$$

$$\mathfrak{r}(\mathbf{x}) = g^{ij}(\mathbf{x}) \text{Ricci}_{ij}(\mathbf{x}) \quad (1.6)$$

Lemma 1.1 *If \mathbf{x} and \mathbf{y} are Riemann normal coordinate expressions about some point for x and y which are close enough so the distance between them $d(x, y)$*

is defined, then

$$d(x, y)^2 = |\mathbf{x} - \mathbf{y}|^2 + \mathcal{O}\left(|\mathbf{x}|^2 |\mathbf{y}|^2\right). \quad (1.7)$$

Proof: Consider the difference $f(\mathbf{x}, \mathbf{y}) = d(x, y)^2 - |\mathbf{x} - \mathbf{y}|^2$ as a function of \mathbf{x} for fixed \mathbf{y} . It is smooth, and, since the exponential preserves length along geodesics through the origin, $f(0, \mathbf{y}) = 0$, and the directional derivative of f at $\mathbf{x} = 0$ in the direction \mathbf{y} is zero as well. Gauss's lemma ensures the directional derivative at $\mathbf{x} = 0$ in directions perpendicular to \mathbf{y} are also 0. So $\partial f / \partial \mathbf{x} = 0$ at $\mathbf{x} = 0$. Since this is true for all \mathbf{y} , $\partial^2 f / \partial \mathbf{y}^2$ and $\partial^3 f / \partial \mathbf{x} \partial \mathbf{y}^2 = 0$ at $\mathbf{x} = 0$, as well, so $\partial^2 f / \partial \mathbf{y}^2$ is bounded by some $c |\mathbf{x}|^2$ for all x and y by Taylor's theorem. By symmetry, f and $\partial f / \partial \mathbf{y}$ are zero when $\mathbf{y} = 0$, so integrating $\partial^2 f / \partial \mathbf{y}^2$ twice against y gives the result. \square

1.2 HEAT EQUATIONS AND APPROXIMATE KERNELS

Of particular interest are approximate solutions to various versions of the heat equation in Riemann normal coordinates.

Lemma 1.2 *In a neighborhood of 0 in Riemann normal coordinates if*

$$\Delta_0 = \frac{1}{2} g^{ij} (\partial_i \partial_j - \Gamma_{ij}^k \partial_k)$$

and

$$K_0(\mathbf{x}; t) = (2\pi t)^{-n/2} e^{-\frac{\delta_{jk} x^j x^k}{2t} + \frac{1}{12} \text{Ricci}_{kl} x^k x^l + \frac{t\mathbf{x}}{12}}$$

then

$$\frac{\partial K_0}{\partial t} - \Delta_0 K_0 = \mathcal{O}\left(t + |\mathbf{x}| + |\mathbf{x}|^3/t + |\mathbf{x}|^5/t^2\right) K_0. \quad (1.8)$$

Likewise if $f(\mathbf{x})$ is twice differentiable with f and all its first and second derivatives bounded by C and

$$\Delta_1 = \Delta_0 + f(\mathbf{x})$$

and

$$K_1(\mathbf{x}; t) = K_0(\mathbf{x}; t) e^{\frac{t}{2}[f(\mathbf{x}) + f(0)]}$$

then

$$\frac{\partial K_1}{\partial t} - \Delta_1 K_1 = \mathcal{O}\left(t + |\mathbf{x}| + |\mathbf{x}|^3/t + |\mathbf{x}|^5/t^2 + Ct + C|\mathbf{x}|^2 + C^2 t^2\right). \quad (1.9)$$

Finally if h_k^l is for $0 \leq k, l \leq n$ a collection of functions of \mathbf{x} all bounded by D , then defining

$$\Delta_2 = \Delta_1 + h_k^l x^k \partial_l + h_k^k / 2$$

and

$$K_2(\mathbf{x}; t) = K_1(\mathbf{x}; t) e^{-h_k^l \delta_{ll'} x^k x^{l'} / 2}$$

then

$$\begin{aligned} \frac{\partial K_2}{\partial t} - \Delta_2 K_2 = & \mathcal{O}\left(t + |\mathbf{x}| + |\mathbf{x}|^3/t + |\mathbf{x}|^5/t^2 + Ct + C|\mathbf{x}|^2 \right. \\ & \left. + C^2 t^2 + D|x| + D|\mathbf{x}|^3/t + CD|x|t + D^2|x|^2\right) K_2. \end{aligned} \quad (1.10)$$

Proof: Here, and in the remainder of this proof, write x_j for $\delta_{jj'} x^{j'}$. For Eq. (1.8), observe that

$$\frac{\partial K_0}{\partial t} = \left[-\frac{n}{2t} + \frac{x^j x_j}{2t^2} + \frac{\mathfrak{r}}{12} \right] K_0 \quad (1.11)$$

and

$$\partial_i K_0 = \frac{\partial K_0}{\partial x^i} = \left[-\frac{x_i}{t} + \frac{\text{Ricci}_{ik} x^k}{6} + \mathcal{O}\left(t + |\mathbf{x}|^2\right) \right] K_0. \quad (1.12)$$

Therefore (implicitly symmetrizing both sides, since this term occurs in Δ only symmetrically)

$$\begin{aligned} \partial_i \partial_j K_0 = & \left[-\frac{\delta_{ij}}{t} + \frac{\text{Ricci}_{ij}}{6} + \frac{x_i x_j}{t^2} \right. \\ & \left. - \frac{\text{Ricci}_{jk} x^k x_i}{3t} + \mathcal{O}\left(t + |\mathbf{x}| + |\mathbf{x}|^3/t\right) \right] K_0 \end{aligned}$$

and

$$\Gamma_{ij}^k \partial_k K_0 = \left[-\Gamma_{ij}^k \frac{x_k}{t} + \mathcal{O}(|\mathbf{x}|) \right] K_0.$$

Using Eqs. (1.3) and (1.2) to expand the metric and Christoffel symbol in Riemann normal coordinates, and dropping low enough order terms

$$\begin{aligned} \Delta_0 K_0 = & \left[-\frac{g^{ij} \delta_{ij}}{2t} + \frac{g^{ij} \text{Ricci}_{ji}}{12} + \frac{g^{ij} x_i x_j}{2t^2} - \frac{g^{ij} \text{Ricci}_{jk} x^k x_i}{6t} \right. \\ & \left. + \frac{g^{ij} \Gamma_{ij}^k x_k}{2t} + \mathcal{O}\left(t + |\mathbf{x}| + |\mathbf{x}|^3/t\right) \right] K_0 \\ = & \left[-\frac{n}{2t} + \frac{R_{kil}^i x^k x^l}{6t} + \frac{\mathfrak{r}}{12} + \frac{x^k x_k}{2t^2} - \frac{\text{Ricci}_{kl} x^k x^l}{6t} + \frac{\delta^{ii'} \Gamma_{ii'}^k x_k}{2t} \right. \\ & \left. + \mathcal{O}\left(t + |\mathbf{x}| + |\mathbf{x}|^3/t + |\mathbf{x}|^5/t^2\right) \right] K_0 \\ = & \left[-\frac{n}{2t} + \frac{x^k x_k}{2t^2} + \frac{\mathfrak{r}}{12} + \mathcal{O}\left(t + |\mathbf{x}| + |\mathbf{x}|^3/t + |\mathbf{x}|^5/t^2\right) \right] K_0 \end{aligned} \quad (1.13)$$

where we have used Eqs. (1.3) and (1.5) in the last line. Subtracting Eqs. (1.11) and (1.13) gives the result.

For Eq. (1.9), notice $\partial K_1/\partial t$ is given by all the terms in Eq. (1.11) multiplied by K_1 instead of K_0 plus in addition

$$\frac{f(0) + f(\mathbf{x})}{2} K_1.$$

Also

$$\partial_i e^{\frac{t}{2}[f(0)+f(\mathbf{x})]} = \frac{t}{2} \frac{\partial f(\mathbf{x})}{\partial x^i} e^{\frac{t}{2}[f(0)+f(\mathbf{x})]}$$

so estimating $\partial_i \partial_j e^{\frac{t}{2}[f(0)+f(\mathbf{x})]}$ as $\mathcal{O}(Ct + C^2 t^2)$ and estimating $\partial_i K_0 \partial_j e^{\frac{t}{2}[f(0)+f(\mathbf{x})]}$ with the help of Eq. (1.12) we see $\Delta_1 K_1$ contains all the terms in Eq. (1.13) (times K_1 instead of K_0) plus

$$\left[f(\mathbf{x}) - \frac{x^i}{2} \frac{\partial f(\mathbf{x})}{\partial x^i} + \mathcal{O}(tC + t^2 C^2) \right] K_1.$$

Noticing that

$$\frac{f(0) + f(\mathbf{x})}{2} = f(\mathbf{x}) - \frac{x^i}{2} \frac{\partial f(\mathbf{x})}{\partial x^i} + \mathcal{O}(C|\mathbf{x}|^2)$$

by Taylor's Theorem gives the result.

For Eq. (1.10), there are no additional terms to $\partial K_2/\partial t$ beyond those in $\partial K_1/\partial t$ except multiplied by K_2 instead of K_1 . The additional term in $\partial_j K_2$ is (writing h_{jk} for $\delta_{jj'} h_k^{j'}$)

$$\left(-\frac{1}{2} [h_{jk} + h_{kj}] x^k + D |\mathbf{x}|^2 \right) K_2$$

So the additional terms in $\partial_i \partial_j K_2$ are (symmetrizing on i and j)

$$\left[\frac{h_{kj} x^k x_i + h_{jk} x^k x_i}{t} - h_{ij} + \mathcal{O}(D|\mathbf{x}| + CDt|\mathbf{x}|^2 + D^2|\mathbf{x}|^4) \right] K_2$$

which means that the additional terms in $\Delta_2 K_2$ consist of

$$\left[\frac{g^{ij} h_{jk} x^k x_i}{t} - \frac{1}{2} g^{ij} h_{ij} - h_{kl} x^k \frac{x^l}{t} + h_k^k/2 \right. \\ \left. + \mathcal{O}(D|\mathbf{x}| + D|\mathbf{x}|^3/t + CD|\mathbf{x}|^2 t + D^2|\mathbf{x}|^4) \right] K_2$$

from which the result follows. \square

2 Grassman Variables and the Norm

The first subsection on Grassman variables is completely standard and can be found for example in [MQ86, Rog92a]. Expressing operators on forms in terms of kernels in Grassman variable is also in [Rog92a]. The norm $|\cdot|_t$ is designed primarily to keep track of the error terms in the large partition limit in such a fashion that the supertrace of the error terms goes to zero as required for Gauss-Bonnet-Chern theorem. Apart from various baroque analytic additions, the idea of the norm and the rank of terms is a fairly natural scaling wherein the spatial variable \mathbf{y}_x scales as $t^{1/2}$ and the Grassman variable ψ scales as $t^{-1/2}$. Lemma 2.1 collects facts about the norm used elsewhere, but the key fact is the first, which uses the presence of the Gaussian to translate powers of the spatial variable into powers of $t^{1/2}$ near the diagonal, and an exponentially damped term away from the diagonal. This is crucial to the proof of Proposition 3.1.

2.1 GRASSMAN VARIABLES

If V is a vector space with basis v_1, \dots, v_n then a k -linear function of k variables $w_i = w_i^j v_j$ for $1 \leq i \leq k$ takes the form

$$f(w_1, \dots, w_k) = C_{j_1, j_2, \dots, j_k} w_1^{j_1} w_2^{j_2} \dots w_k^{j_k}.$$

The antisymmetrization of this multilinear function is the element of $\Lambda^* V$ given by

$$Af(v_1, \dots, v_k) = \sum_{\sigma} \frac{(-1)^{\sigma}}{k!} f(w_{\sigma(1)}, \dots, w_{\sigma(k)})$$

which is again f if it was already antisymmetric. This map from multilinear functions to antisymmetric ones is manifestly basis independent, but using the basis gives a simple representation of $\Lambda^* V$ as the algebra generated freely by $\psi^1, \psi^2, \dots, \psi^n$ subject only to the antisymmetry relation $\psi^j \psi^j = -\delta \psi^j \psi^i$. The antisymmetry map is then given naturally by writing $\psi = \psi^1 + \dots + \psi^n$ and interpreting Af as

$$f(\psi) \equiv C_{j_1, j_2, \dots, j_k} \psi^{j_1} \dots \psi^{j_k}.$$

Call such an f an antisymmetric polynomial in the Grassman variable ψ with values in V .

To represent endomorphisms on $\Lambda^* V$, define maps $\partial/\partial\psi^j$ for $1 \leq j \leq n$ by

$$\frac{\partial}{\partial\psi^j} \psi^j \psi^L = \psi^L \quad \frac{\partial}{\partial\psi^j} \psi^L = 0$$

if $\psi^L = \psi^{l_1} \dots \psi^{l_k}$ and $j \neq l_i$ for all i . It is easy to confirm that multiplication operators ψ^j and the differentiation operators $\partial/\partial\psi^j$ generate all endomorphisms. In fact the maps $\partial/\partial\psi^j$ generate a free antisymmetric algebra as well, so the same convention applies to interpret multilinear functions $g(\partial/\partial\psi)$ of these. Expressions of the form $f(\psi) g(\partial/\partial\psi)$ define endomorphisms (and in fact span all endomorphisms). Operators that preserve form degree are spanned by those products in which f and g are homogeneous of the same monomial degree. Filter

the form-degree-preserving endomorphisms by this common monomial degree, and call this the degree of the endomorphism. An endomorphism of degree p is zero on all elements of Λ^*V of degree less than p and nonzero on at least one element of degree p .

Given a choice of degree n element of Λ^*V , any operator \mathfrak{k} on Λ^*V determines a kernel; that is, an antisymmetric polynomial $K(\psi_x, \psi_y)$ in two Grassman variables ψ_x, ψ_y valued in V (which of course anticommute with each other). Explicitly, applying \mathfrak{k} to $f(\psi)$ is

$$(\mathfrak{k}f)(\psi) = \oint K(\psi, \psi_y) f(\psi_y) d\psi_y$$

where the Berezin integral $\oint f(\psi_y) d\psi_y$ returns the coefficient of the chosen top element of Λ^*V in f . Such kernels in turn admit a sort of Fourier transform, in terms of a Grassman variable ρ valued in V^* and an antisymmetric polynomial in three variables $k(\psi_x, \rho, \psi_y)$:

$$K(\psi_x, \psi_y) = \oint e^{i\langle \rho, \psi_x - \psi_y \rangle} k(\psi_x, \rho, \psi_y) d\rho$$

where the Berezin integral is with respect to the element of ΛV dual to the chosen element of Λ^*V . If $k(\psi_x, \rho, \psi_y)$ is of the form $f(\psi_x)g(\rho)h(\psi_y)$, the operator \mathfrak{k} is simply $\mathfrak{k} = f(\psi)g(i\partial/\partial\psi)h(\psi)$. Degree-preserving endomorphisms are spanned by homogeneous k in which the degree of ρ equals the sum of degrees of ψ_x and ψ_y , and the degree of the endomorphism is the degree of ρ .

2.2 BUNDLES

With the help of the mild subterfuge of the previous subsection, sections of all bundles in this paper may be expressed using the “function of variables” notation familiar from freshman calculus. Usually these sections will also depend on a positive real parameter t (or sometimes t_j or τ_j), which will appear in the list of parameters at the end, after a semicolon, to emphasize its special role. For the entire paper except Corollary 6.1 this t will be bounded above by an arbitrary constant, and all other constants, explicit and implicit, will depend on this bound as well as on the manifold M .

For instance, $f(x, \psi_x)$ and $f(x, \psi_x; t)$ represent sections of the bundle Λ^*T_M over M , i.e. elements of $\Gamma(\Lambda^*T_M)$, or forms. This notation is meant to indicate that at each x , ψ_x is a Grassman variable taking values in T_xM , and $f(x, \psi_x)$ is a multilinear function of this variable which thus represents an element of Λ^*T_xM . One useful aspect of this particular notation is that, assuming the Berezin integral uses the measure coming from the metric on T_xM , $\int \oint f(x, \psi_x) d\psi_x dx$ represents the integral of the top form component of f , regardless of the metric. That is, it represents the usual integral on forms.

The arguments in the paper are concerned with sections of a particular bundle over $(x, y) \in M \times M$ which will be most conveniently discussed in two distinct representations. (In fact, the representations only agree near the diagonal, but that suffices) The first bundle is $\Lambda^*T_xM \otimes \Lambda^*T_yM$, whose sections in

$\Gamma(\Lambda^*T_xM) \otimes \Gamma(\Lambda^*T_yM)$ can be represented by functions $L(x, y; \psi_x, \psi_y)$ in the obvious fashion. Define $E_{x,y}$ to be the set of all such sections which are degree preserving (that is, the kernel is a sum of monomials in which the degree of ψ_x and the degree of ψ_y add to n). As in the previous subsection, such sections represent kernels of operators on forms, via the $*$ product

$$(\mathfrak{I}f)(x, \psi_x) = [L * f](x, \psi_x) = \int \oint L(x, y, \psi_x, \psi_y) f(y, \psi_y) d\psi_y dy. \quad (2.1)$$

Usually the kernels will only be nonzero when x and y are within the injectivity radius of each other, so that the parallel transport map \mathfrak{P}_y^x from T_xM to T_yM is defined. Naturally associated to this map (via transpose and antisymmetric product) is the map Λ^*T_yM to Λ^*T_xM , also called parallel transport \mathfrak{P}_y^x . Such kernels take the form

$$\oint e^{i\langle \rho, \mathfrak{P}_y^x \psi_x - \psi_y \rangle} k(x, y, \psi_x, \psi_y, \rho) d\rho.$$

The simplest example would be the kernel

$$P(x, y, \psi_x \psi_y) = \oint e^{i\langle \rho, \mathfrak{P}_y^x \psi_x - \psi_y \rangle} d\rho$$

($P = 0$ when \mathfrak{P}_y^x is not defined) which is the kernel of the parallel transport operator itself, always assumed to be along a minimal geodesic.

If x and y are sufficiently close, then $L \in E_{x,y}$ determines a section $\overline{L} \in \overline{E}_{x,y}$ of the bundle $\text{End}_0(\Lambda^*T_xM) \otimes \mathbb{R}_y$ defined by

$$L(x, y, \psi_x, \psi_y) = \overline{L}(x, y) P(x, y, \psi_x, \psi_y).$$

Here the subscript 0 is a reminder that the endomorphisms are degree-preserving, and the subscript on \mathbb{R}_y indicates the bundle is only over the first factor of M .

2.3 THE NORM ON THE BUNDLE

Fix for the rest of the paper an arbitrary ϵ with $1/2 > \epsilon > 0$. All explicit and implicit constants in what follows will depend on ϵ . Define a norm $|\cdot|_t$ depending on a positive real t on $\text{End}_0(T_xM)$ as follows:

Definition 1 *In terms of an orthonormal basis,*

$$\left| \psi^{j_1} \dots \psi^{j_k} \frac{\partial}{\partial \psi^{l_1}} \dots \frac{\partial}{\partial \psi^{l_k}} \right|_t = \begin{cases} 1 & k \leq 2 \\ t^{(k-2)(-1/2+\epsilon/n)} & k \geq 2. \end{cases} \quad (2.2)$$

This is the inner product norm times a power of t depending only on the degree. If $\overline{L} \in \overline{E}_{x,y}$ define $|\overline{L}|_t$ to be the function of x and y given by this norm on each fiber, and for $L \in E_{x,y}$ define $|L|_t = |\overline{L}|_t$.

There is a real-valued kernel H of particular importance, because, where defined, it agrees with the flat-space heat operator; namely

$$H(x, y; t) = (2\pi t)^{-n/2} e^{-\frac{|\mathbf{x}_y|^2}{2t}} \quad (2.3)$$

if x and y are within the injectivity radius of each other and $H = 0$ otherwise. The errors introduced by using approximate heat kernels on forms will generally be t -dependent sections which take the form of H times a section well-behaved as t goes to zero.

Suppose $L \in E_{x,y}$ is a t -dependent kernel of degree k , and suppose there are real numbers $C > 0$, r , and s such that $|L| \leq Ct^r |\mathbf{y}_x|^s$ for all x and y within the injectivity radius. Then define L to be of rank p where $p = \min(2r + s, 2r + s + 2 - k)$. The sum of terms of rank p of different degrees is still called rank p . The same definition applies to elements of $\overline{E}_{x,y}$ and the rank of L is the rank of \overline{L} .

For instance $\text{Ricci}(\mathbf{y}_x, \mathbf{y}_x)$, $t\tau$, and $t(\rho, R[\psi_x, \psi_x]\rho)$ all are rank 2, while $\langle \rho, R[\mathbf{y}_x, \psi_x]\mathbf{y}_x \rangle$ is rank 3. The significance of rank is explained in the next lemma:

Lemma 2.1

- (a) If $\overline{L} \in \overline{E}_{x,y}$ is of rank p or less and is a smooth function of t times t^c , where $c \geq -2$, there is an $A > 0$ such that

$$\overline{L}H = \overline{F}_1H + \overline{F}_2t \quad (2.4)$$

where $\overline{F}_i \in \overline{E}_{x,y}$ and $|\overline{F}_i|_t \leq At^{(p-1)/2+\epsilon}$.

- (b) If \overline{L} is a smooth function of $t_1 \leq t$ of degree at most 1 and rank 2, or degree 2 and rank 2 by virtue of being a multiple of t_1 , then there are $B, D > 0$ such that if $\overline{F} \in \overline{E}_{x,y}$

$$\begin{aligned} |\overline{L}H(t_1)\overline{F}|_t &\leq Bt_1^{\epsilon/n} |\overline{F}|_t [H(t_1) + Dt_1] \\ |\overline{F}\overline{L}H(t_1)|_t &\leq Bt_1^{\epsilon/n} |\overline{F}|_t [H(t_1) + Dt_1]. \end{aligned} \quad (2.5)$$

- (c) If $t < t'$ then $|F|_{t'} \leq |F|_t$.

- (d) There is a $C > 0$ such that for all $\overline{F}, \overline{G} \in \overline{E}_{x,y}$

$$|\overline{F}\overline{G}|_t \leq Ct^{-1} |\overline{F}|_t |\overline{G}|_t. \quad (2.6)$$

Proof:

- (a) It suffices to prove the result for \overline{L} of rank exactly p , and degree $k \geq 0$. The proof of Eqs. (2.4) and (2.5), as well as of Eq. (4.2) below, rely on the following trick for converting powers of \mathbf{y}_x into powers of t in the presence of the Gaussian H . Choose $\delta, C > 0$, and consider separately the case $|\mathbf{y}_x| > Ct^{1/2-\delta}$ and $|\mathbf{y}_x| \leq Ct^{1/2-\delta}$. In the former case $H(x, y; t) = \mathcal{O}(t^{-n/2}) e^{C^2 t^{-2\delta}/2}$. This is exponentially damped, and thus can be bounded by some multiple of any power of t desired; in particular, $\overline{L}H = \overline{F}_2t$ with \overline{F}_2 bounded as in the statement of Eq. (2.4).

In the case $|\mathbf{y}_x| \leq Ct^{1/2-\delta}$, since \overline{L} is rank p , it is bounded by a multiple of $t^r |\mathbf{y}_x|^s$ where $p = \min(2r+s, 2r+s+2-k)$. The bound on $|\mathbf{y}_x|$ implies \overline{L} is bounded by a multiple of $t^{(2r+s)/2-s\delta}$. Thus $|\overline{L}|_t$ is bounded by a multiple of $t^{p/2-s\delta}$. By assumption, $r \geq -2$, so s is bounded (by $p+4$ if $k \leq 2$ and by $p+k+2$ otherwise), and therefore, given $1/2 > \epsilon > 0$, δ can be chosen small enough that in fact $|\overline{L}|_t$ is bounded by a multiple of $t^{(p-1)/2+\epsilon}$.

- (b) First notice multiplying two elements of $\text{End}_0(\Lambda^* TM)$ of fixed degree together gives a product whose degree is the sum of their degrees, so that multiplying a degree k term (smooth in t) times \overline{F} multiplies the t -norm by a multiple of $t^{-k(1/2-\epsilon/n)}$.

Again the result is easy if $|\mathbf{y}_x| > Ct_1^{1/2-\delta}$, for some C and δ . If instead $|\mathbf{y}_x| \leq Ct_1^{1/2-\delta}$, then in the case that \overline{L} is degree at most 1 and rank 2, its smoothness implies $r \geq 0$ so it is bounded by a multiple of $t_1^{1-2\delta}$. Under multiplication it thus multiplies the t -norm of F by a constant times $t_1^{1-2\delta} t^{-1/2+\epsilon/n} \leq t_1^{1/2+\epsilon/n-2\delta}$, which with the right choice of δ is bounded by a multiple of $t_1^{\epsilon/n}$. In the case where \overline{L} is a multiple of t_1 times a degree 2 term, then upon multiplication it multiplies the t norm of \overline{F} by a multiple of $t_1 t^{2(-1/2+\epsilon/n)} \leq t_1^{\epsilon/n}$.

- (c) Trivial

- (d) Multiplication in $\text{End}_0(\Lambda^* TM)$ adds degree, and clearly is bounded with respect to any fixed norm. If the degrees k and j are both at least 2, then the t -norm of the product is a multiple of $t^{(k+j-2)(-1/2+\epsilon/n)}$ which is a multiple of $t^{2(-1/2+\epsilon/n)}$ times the t -norms of the factors, so in particular a multiple of t^{-1} times the product of the t -norms. If either degree is less than 2, the same is true with a higher power of t , and therefore in general it is a multiple of t^{-1} times the product of t -norms.

□

2.4 THE SUPERTRACE

The matrix supertrace str of an operator $\mathfrak{k} \in \text{End}_0(\Lambda^* V)$ is the trace of the restriction of \mathfrak{k} to even forms minus the trace of its restriction to odd forms. In terms of Grassman variables, the kernel of \mathfrak{k} is $K(\psi_x, \psi_y) = K_{IJ} \psi_x^I \psi_y^J$ and the trace of its restriction to forms of degree d is, in an orthonormal basis,

$$\text{tr } \mathfrak{k}|_d = \sum_{|I|=d} K_{I\bar{I}} (-1)^{\bar{I}I},$$

where \bar{I} is formed from the complement of I in $\{1, \dots, n\}$, and the last factor is the sign of the permutation. This follows from noticing $(-1)^{\bar{I}I} \psi^{\bar{I}}$ is the dual

basis vector to ψ^I . On the other hand

$$\oint K(\psi, \psi) d\psi = \sum_{IJ} K_{IJ} \oint \psi^I \psi^J d\psi$$

The integrals on the right vanish, unless $J = \bar{I}$, so the sum reduces to $\sum_I K_{I\bar{I}}(-1)^{I\bar{I}}$. Since $(-1)^{I\bar{I}} = (-1)^{|I||\bar{I}|}(-1)^{\bar{I}I}$, decomposing this sum according to degree and comparing with the sum for $\text{tr } \mathfrak{k}|_d$, shows

$$\oint K(\psi, \psi) d\psi = \text{str } \mathfrak{k} \quad (2.7)$$

since n is even.

Lemma 2.2 *If F is the kernel of an operator in $\text{End}_0(\Lambda^*V)$ and $|F|_t \in \mathcal{O}(t^{p+1+\epsilon})$, then*

$$\text{str}(F) \in \mathcal{O}(t^{p+n/2+2\epsilon/n}).$$

Proof: From the above characterization of the supertrace, $F = \text{str}(F)\psi^1 \cdots \psi^n \frac{\partial}{\partial \psi^1} \cdots \frac{\partial}{\partial \psi^n}$ plus lower-degree terms.

The condition on F and the definition in Eq 2.2 of the norm on $\text{End}_0(\Lambda^*V)$ thus require

$$\text{str}(F)t^{(n-2)(-1/2+\epsilon/n)} = \text{str}(F) \left| \psi^1 \cdots \psi^n \frac{\partial}{\partial \psi^1} \cdots \frac{\partial}{\partial \psi^n} \right|_t \leq |F|_t = \mathcal{O}(t^{p+1+\epsilon})$$

from which the lemma immediately follows. \square

3 The Approximate Kernel

For background on the heat operator see [BGV04]. Proposition 3.1, which says K is an approximate solution to the heat equation with error terms that are Gaussians times errors bounded in the t -norm, is entirely computational.

3.1 THE APPROXIMATE KERNEL

The small t approximation to the propagator of $N = 1$ imaginary time supersymmetric quantum mechanics in curved space is given by the kernel

$$\begin{aligned} K(x, y, \psi_x, \psi_y; t) = & \oint H \exp \left[-\frac{t\mathfrak{r}}{6} + \frac{1}{12} \text{Ricci}(\mathbf{x}_y, \mathbf{x}_y) \right. \\ & \left. + i \langle \rho, \mathfrak{P}_y^x \psi_x - \psi_y \rangle + \frac{t}{8} (\rho, R[\psi_y, \psi_y] \rho) + \frac{t}{8} (\mathfrak{P}_y^x \rho, R[\psi_x, \psi_x] \mathfrak{P}_y^x \rho) \right] d\rho \end{aligned} \quad (3.1)$$

when x and y are close enough to be within the injectivity radius of each other (so K is invertible in this region) and zero otherwise. All curvature tensors are understood to be evaluated at y except for the last R which is evaluated at x .

The analytic properties of K crucial in what follows will also hold for any kernel K' which differs from K by a term which in Riemann normal coordinates around y (that is in terms of \mathbf{x}_y) is $H(x, y; t)$ times a term of rank 3 (see §2.3). Accordingly, write $K \sim K'$ if the two kernels differ by such terms.

Observe that $K = (P + L)H$, where P is the parallel transport operator, and L is a sum of terms each of which has the form a positive power of t , times a power of \mathbf{x}_y , times a product of ψ_x, ρ, ψ_y , wherein the sum of the power of \mathbf{x}_y and twice the power of t is at least the degree of the $\psi\rho$ factors. As a consequence, L times any term of rank p is again rank p . Since P does not affect rank, any rank 3 term times K is equal to some rank 3 term times H .

For example the Ricci and \mathfrak{r} terms are of rank 2, and replacing them with the same terms interpreted at x (with \mathbf{y}_x replacing \mathbf{x}_y), changes K by terms of rank 3 times H (since Ricci and \mathfrak{r} are both continuously differentiable). Thus this change produces a kernel equivalent to K under \sim . Likewise, changing variables from ρ to $\mathfrak{P}_y^x \rho$ (since \mathfrak{P} is measure-preserving there is no Jacobian factor) shows that, up to H times terms of rank 3, K is symmetric with respect to x and y .

In fact, up to such terms, K has a simple expression in Riemann normal coordinates centered at y . Notice

$$\langle \rho, \mathfrak{P}_y^x \psi_x - \psi_y \rangle = \langle \rho, \psi_x - \psi_y \rangle + \frac{1}{6} \langle \rho, R[\mathbf{x}_y, \psi_x] \mathbf{x}_y \rangle + \mathcal{O}(|\mathbf{y}_x|^3 \psi_y^i \rho_j)$$

the error term being rank 3. Likewise, in the term with $\mathfrak{P}_y^x \rho$, the \mathfrak{P}_y^x can be removed at the cost of a rank 4 term, and R_x can be replaced with R_y at the cost of a rank 3 term. This leads to

$$\begin{aligned} K \sim \oint H \exp \left[i \langle \rho, \psi_x - \psi_y \rangle - \frac{t\mathfrak{r}}{6} + \frac{1}{12} \text{Ricci}(\mathbf{x}_y, \mathbf{x}_y) + \frac{i}{6} \langle \rho, R[\mathbf{x}_y, \psi_x] \mathbf{x}_y \rangle \right. \\ \left. + \frac{t}{8} (\rho, R[\psi_x, \psi_x] \rho) + \frac{t}{8} (\rho, R[\psi_y, \psi_y] \rho) \right] d\rho. \end{aligned} \quad (3.2)$$

That is, defining K' as

$$\begin{aligned} K' \equiv \oint H \exp \left[i \langle \rho, \psi_x - \psi_y \rangle - \frac{t\mathfrak{r}}{6} + \frac{1}{12} \text{Ricci}(\mathbf{x}_y, \mathbf{x}_y) + \frac{i}{6} \langle \rho, R[\mathbf{x}_y, \psi_x] \mathbf{x}_y \rangle \right. \\ \left. + \frac{t}{8} (\rho, R[\psi_x, \psi_x] \rho) + \frac{t}{8} (\rho, R[\psi_y, \psi_y] \rho) \right] d\rho, \end{aligned} \quad (3.3)$$

$$K \sim K'.$$

3.2 THE HEAT OPERATOR

Let Δ_{LdR} be the operator on forms given by

$$\Delta_{\text{LdR}} = -g^{ij} (\nabla_i \nabla_j - \Gamma_{ij}^k \nabla_k) + \frac{1}{2} \left(\frac{\partial}{\partial \psi}, R[\psi, \psi] \frac{\partial}{\partial \psi} \right) \quad (3.4)$$

where $\Delta_i = \partial_i - \Gamma_{ij}^k \psi^j \partial / \partial \psi^k$.

Proposition 3.1 For $K = K(x, y, \psi_x, \psi_y; t)$ defined by Eq. (3.1) and Δ by Eq. (3.4)

$$\begin{aligned} \frac{\partial K}{\partial t} + \frac{1}{2} \Delta_x K &= F_1 H(x, y; t) + F_2 t, \text{ and} \\ \frac{\partial K}{\partial t} + \frac{1}{2} \Delta_y K &= F_3 H(x, y; t) + F_4 t, \end{aligned} \quad (3.5)$$

where $F_i \in E_{x,y}$ and $|F_i|_t = \mathcal{O}(t^\epsilon)$.

Proof:

Recalling Eq. (2.4) of Lemma 2.1, the argument reduces to showing that the heat operators on the left-hand sides applied to K give terms of rank 1 times H . To this end, first observe that the heat operator acts on H times terms of a given rank to produce H times terms whose rank is lower by at most 2. This is because the operator $\partial/\partial t$ lowers the rank of a smooth function of t by 2, and, when applied to H , gives $H\delta_{ij}x^jx^j/2t^2$, which also lowers the rank of whatever it multiplies by 2. In Δ_x , the curvature term does not change rank at all (and is zero on H). The operator ∇_i in the other hand, lowers rank by one for smooth terms because ∂_i lowers the degree of \mathbf{x}_y by 1 and the $\Gamma_{ij}^k\psi^j\partial/\partial\psi^k$ actually increases rank by 1 (by Eq. (1.3)) Acting on H , ∇_i gives $-H\delta_{ij}x^j/t$, which lowers the rank of whatever it multiplies by 1. Thus, taken together, the heat operator will lower rank by at most two. As a first consequence, if the heat operator applied to some $\hat{K} \sim K$ gives terms of rank 1 times H , then the same is true for K itself. This in turn implies the two equations in the proposition are equivalent, since, as noted above, K is symmetric up to terms of rank 3 times H .

It suffices, then, to confirm the result on $K' \sim K$, where K' is defined in Riemann normal coordinates by the right-hand side of Eq. (3.3). In fact, simplifying the heat operator itself in Riemann normal coordinates, by dropping terms that when applied to K' give H times terms of rank 1 or higher, will lead to an operator Δ_2 and a kernel $K_2 \sim K'$ for which, by Eq. (1.10) of Lemma 1.2, the proposition holds.

Begin by expanding Δ_{LdR} in Riemann normal coordinates,

$$\begin{aligned} \Delta_{\text{LdR}} &= -g^{ij}\partial_i\partial_j + 2g^{il}\Gamma_{iq}^p\psi^q\frac{\partial}{\partial\psi^p}\partial_l + g^{il}\partial_l(\Gamma_{iq}^p)\psi^q\frac{\partial}{\partial\psi^p} - g^{il}\Gamma_{iq}^p\Gamma_{lk}^n\psi^q\frac{\partial}{\partial\psi^p}\psi^k\frac{\partial}{\partial\psi^n} \\ &\quad + g^{il}\Gamma_{il}^q\partial_q - g^{il}\Gamma_{il}^q\Gamma_{qp}^k\psi^p\frac{\partial}{\partial\psi^k} + \frac{1}{2}\left(\frac{\partial}{\partial\psi}, R[\psi, \psi]\frac{\partial}{\partial\psi}\right) \\ &\sim -2\Delta_0 + 2\delta^{il}\Gamma_{iq}^p\psi^q\frac{\partial}{\partial\psi^p}\partial_l + \delta^{il}\partial_l(\Gamma_{iq}^p)\psi^q\frac{\partial}{\partial\psi^p} + \frac{1}{2}\left(\frac{\partial}{\partial\psi}, R[\psi, \psi]\frac{\partial}{\partial\psi}\right) \\ &\sim -2\Delta_0 - \frac{2}{3}\delta^{il}\left(R_{ikq}^p + R_{qki}^p\right)x^k\psi^q\frac{\partial}{\partial\psi^p}\partial_l + \frac{1}{3}\delta^{ik}R_{qki}^p\psi^q\frac{\partial}{\partial\psi^p} + \frac{1}{2}\left(\frac{\partial}{\partial\psi}, R[\psi, \psi]\frac{\partial}{\partial\psi}\right) \end{aligned}$$

where Δ_0 is defined in Lemma 1.2. The terms omitted, when applied to K' , give terms of rank 1 or higher times H (each ∂_l lowers rank by 1, each factor of Γ increases rank by 1).

Thus

$$\Delta_{\text{LdR}} = -2[\Delta_0 + h_k^l x^k \partial_l + h_k^k/2 + f(\mathbf{x})]$$

where $h_k^l = \frac{1}{3}\delta^{il}(R_{ikq}^p + R_{qki}^p)\psi^q \frac{\partial}{\partial \psi^p}$ and $f = -\frac{1}{4}\left(\frac{\partial}{\partial \psi}, R[\psi, \psi] \frac{\partial}{\partial \psi}\right)$. The expression in Riemann normal coordinates denotes a differential operator on \mathbb{R}^n with coefficients in $\text{End}_0(\Lambda^*\mathbb{R}^n)$, so it still makes sense to refer to the degree and rank of a term. Of course, this is a noncommutative algebra, but since $e^{-h_m^k x^m x^k/2}$ and e^{-tf} commute up to terms of rank 4, Eq. (1.10) applies, with C of degree 2 and D of degree 1, to conclude

$$\frac{\partial K_2}{\partial t} + \frac{1}{2}\Delta_{\text{LdR}}K_2 = FK_2$$

where

$$K_2(\mathbf{x}; t) = He^{\frac{1}{12}\text{Ricci}_{ij}x^i x^j + \frac{t\mathbf{r}}{12} - \frac{1}{6}R_{kij}^l x^i x^j \psi^k \frac{\partial}{\partial \psi^l} - \frac{t}{4}\left(\frac{\partial}{\partial \psi}, \overline{R}[\psi, \psi] \frac{\partial}{\partial \psi}\right)},$$

$\overline{R} = \frac{1}{2}(R_x + R_y)$, and F , representing all the contributions to the error term in Eq. (1.10), is of rank 1.

It remains to show $K_2 \sim K'$, when the first is expressed as a kernel integrated over ρ and the second is given by Eq. (3.3).

Towards that goal note

$$\begin{aligned} 2\left(\frac{\partial}{\partial \psi}, \overline{R}[\psi, \psi] \frac{\partial}{\partial \psi}\right) &= R_{ijkl}(x) \frac{\partial}{\partial \psi^l} \psi^i \psi^j \frac{\partial}{\partial \psi^k} + R_{ijkl}(y) \frac{\partial}{\partial \psi^l} \psi^i \psi^j \frac{\partial}{\partial \psi^k} \\ &= -R_{ijkl}(x) \psi^i \psi^j \frac{\partial}{\partial \psi^k} \frac{\partial}{\partial \psi^l} + R_{ijkil}(x) \psi^j \frac{\partial}{\partial \psi^k} - R_{ijkjl}(x) \psi^i \frac{\partial}{\partial \psi^k} \\ &\quad - R_{ijkl}(y) \frac{\partial}{\partial \psi^k} \frac{\partial}{\partial \psi^l} \psi^i \psi^j + R_{ijjil}(y) \frac{\partial}{\partial \psi^l} \psi^i - R_{ijjil}(y) \frac{\partial}{\partial \psi^l} \psi^j \\ &= -R_{ijkl}(x) \psi^i \psi^j \frac{\partial}{\partial \psi^k} \frac{\partial}{\partial \psi^l} - R_{ijkl}(y) \frac{\partial}{\partial \psi^k} \frac{\partial}{\partial \psi^l} \psi^i \psi^j + 2R_{ijkil}(x) \psi^j \frac{\partial}{\partial \psi^k} + 2R_{ijjil}(y) \frac{\partial}{\partial \psi^l} \psi^j \\ &= -R_{ijkl}(x) \psi^i \psi^j \frac{\partial}{\partial \psi^l} \frac{\partial}{\partial \psi^k} - R_{ijkl}(y) \frac{\partial}{\partial \psi^k} \frac{\partial}{\partial \psi^l} \psi^i \psi^j + 2\mathbf{r}(y) + \text{Rank } 1. \end{aligned}$$

Multiplying by $-t/8$, exponentiating and writing the resulting operator as a kernel integrated over ρ gives

$$He^{-\frac{t}{4}\left(\frac{\partial}{\partial \psi}, \overline{R}[\psi, \psi] \frac{\partial}{\partial \psi}\right)} \sim \oint He^{i\langle \rho, \psi_x - \psi_y \rangle - \frac{t\mathbf{r}}{4} + \frac{t}{8}[(\rho, R[\psi_x, \psi_x]\rho) + (\rho, R[\psi_y, \psi_y]\rho)]} d\rho.$$

The rank 1 error term disappears, since the exponential of t times a rank 1 term is 1 plus a rank 3 term. Combining this with the other Grassman term from K_2 , gives, as an integral over ρ ,

$$\begin{aligned} &He^{-\frac{1}{6}R_{kmjl}x^m x^j \psi^k \frac{\partial}{\partial \psi^l} - \frac{t}{4}\left(\frac{\partial}{\partial \psi}, \overline{R}[\psi, \psi] \frac{\partial}{\partial \psi}\right)} \\ &= \oint He^{i\langle \rho, \psi_x - \psi_y \rangle - \frac{t\mathbf{r}}{4} - \frac{t}{6}R_{kmjl}x^m x^j \rho^l \psi_x^k + \frac{t}{8}[(\rho, R[\psi_x, \psi_x]\rho) + (\rho, R[\psi_y, \psi_y]\rho)]} d\rho. \end{aligned}$$

Comparing the definition of K_2 above with that of K' in Eq. (3.3) gives $K_2 \sim K'$.
 \square

4 The $*$ Product of Kernels and the Norm

The material in this section relies heavily on §2.2 and §2.3, requires the definition Eq. (3.1) of K , and makes one brief reference to Eq. (1.7). Lemmas 4.1 and 4.2 involve rather technical bounds, and are only necessary for the proofs of Propositions 4.1 and 4.2. Only Eqs. (4.6), (4.7), and (4.9) are needed going forward for the rest of the paper. Notice the norm $\|\cdot\|_t$ could not be defined until Lemmas 4.1 and 4.2 guarantee the existence the quantity D in the definition so that Eq. (4.6) holds. Once the norm is defined, Corollary 4.1 can be viewed as the correct statement of Proposition 3.1.

4.1 BOUNDS ON K AND H

Recall that $L_1, L_2 \in E_{x,y}$ represent operators on $\Gamma(\Lambda^*TM)$ and the product of operators corresponds to the $*$ product

$$L_1 * L_2(x, z, \psi_x, \psi_z) = \int \oint L_1(x, y, \psi_x, \psi_y) L_2(y, z, \psi_y, \psi_z) d\psi_y dy.$$

Lemma 4.1 *For bounded $t > 0$, and H as in Eq. (2.3),*

$$\int H(x, y; t) dy = 1 + \mathcal{O}(t); \quad (4.1)$$

for bounded $t_1, t_2 > 0$ with $t = t_1 + t_2$, and ϵ as in §2.3,

$$\begin{aligned} H(t_1) * H(t_2) &= e^{\mathcal{O}(t_1^{\epsilon/n})} [H(t) + \mathcal{O}(t^2)], \\ H(t_1) * H(t_2) &= e^{\mathcal{O}(t_2^{\epsilon/n})} [H(t) + \mathcal{O}(t^2)]. \end{aligned} \quad (4.2)$$

Proof: By Eq. (1.1) the measure $dx = [1 + \mathcal{O}(|\mathbf{x}_y|^2)] d\mathbf{x}_y$ where $d\mathbf{x}_y$ comes from the inner product on T_yM , so Eq. (4.1) becomes

$$\int H(x, y; t) dx = \int (2\pi t)^{-n/2} e^{-\frac{|\mathbf{x}_y|^2}{2t}} [1 + \mathcal{O}(|\mathbf{x}_y|^2)] d\mathbf{x}_y = 1 + \mathcal{O}(t).$$

For Eq. (4.2), following the $t^{1/2-\delta}$ -trick in the proof of Lemma 2.1, consider first the case where $|\mathbf{z}_x|$ is greater than some multiple of $t^{1/2-\delta}$. Then either $|\mathbf{x}_y|$ is more than a multiple of $t_1^{1/2-\delta}$ or $|\mathbf{y}_z|$ is more than a multiple of $t_2^{1/2-\delta}$ so that the entire integral

$$H(t_1) * H(t_2)(x, z) = (2\pi t_1)^{-n/2} (2\pi t_2)^{-n/2} \int e^{-\frac{|\mathbf{x}_y|^2}{2t_1} - \frac{|\mathbf{y}_z|^2}{2t_2}} dy$$

is bounded by a multiple of t^2 times the integral of a multiple of $H(x, y; t)$, which by Eq. (4.1) is bounded by a multiple of t^2 .

If $|\mathbf{z}_x|$ is less than a multiple of $t^{1/2-\delta}$, choose the multiple small enough first of all that x and z are within the injectivity radius of each other. Define

$$E(x, y, z; \alpha) = (1 - \alpha) |\mathbf{x}_y|^2 + \alpha |\mathbf{z}_y|^2 - \alpha(1 - \alpha) |\mathbf{z}_x|^2$$

on the compact domain where $\alpha \in [0, 1]$ and the three lengths are all well-defined, so that if $\alpha = t_1/t$ then

$$\frac{|\mathbf{x}_y|^2}{t_1} + \frac{|\mathbf{z}_y|^2}{t_2} = \frac{|\mathbf{z}_x|^2}{t} + \frac{t}{t_1 t_2} E.$$

Let u be the point on the geodesic between x and z that is α of the way from x to z , so that $\mathbf{u}_x = \alpha \mathbf{z}_x$ and $\mathbf{u}_z = (1 - \alpha) \mathbf{x}_z$. Apply Eq. (1.7), to rewrite $|\mathbf{x}_y|^2$, $|\mathbf{y}_x|^2$ and $|\mathbf{z}_y|^2$ in terms of $|\mathbf{y}_u|$, $|\mathbf{u}_x|$ and $|\mathbf{u}_z|$, and then express these in terms of $|\mathbf{z}_x|$ to obtain $E(x, y, z; \alpha) = |\mathbf{y}_u|^2 \left[1 + \mathcal{O}(\alpha(1 - \alpha) |\mathbf{z}_x|^2) \right]$.

Choosing δ and the coefficient of $t^{1/2-\delta}$ wisely, E is bounded below by $|\mathbf{y}_u|^2 [1 - C\alpha(1 - \alpha)t^\epsilon]$ where C is small enough so this is always positive. Also bound dy by $(1 + D |\mathbf{y}_u|^2) d\mathbf{y}_u$ for some $D > 0$, so

$$\begin{aligned} H(t_1) * H(t_2)(x, z) &= \int (2\pi t_1)^{-n/2} (2\pi t_2)^{-n/2} e^{-\frac{|\mathbf{x}_y|^2}{2t_1} - \frac{|\mathbf{z}_y|^2}{2t_2}} dy \\ &\leq (2\pi t)^{-n/2} e^{-\frac{|\mathbf{z}_x|^2}{2t}} \int (2\pi t_1 t_2 / t)^{-n/2} e^{-\frac{t |\mathbf{y}_u|^2}{2t_1 t_2} [1 - C\alpha(1 - \alpha)t^\epsilon]} [1 + D |\mathbf{y}_u|^2] d\mathbf{y}_u \\ &= H(x, z; t) [1 - C\alpha(1 - \alpha)t^\epsilon]^{-n/2} (1 + D\alpha(1 - \alpha)t^\epsilon)^{n/2}. \end{aligned}$$

From here each estimate in Eq. 4.2 is immediate. In fact, this shows the bound could be strengthened, replacing the exponential factor by $1 + \mathcal{O}(t^\epsilon)$, but the weaker form is sufficient and more convenient later. \square

Lemma 4.2 *For $t > t_1 > 0$ bounded, $K(t_1) = K(x, y, \psi_x, \psi_y; t_1)$ in Eq. (3.1), ϵ and $|\cdot|_t$ as in §2.3, and $\overline{F} \in \overline{E}_{x, y}$,*

$$\begin{aligned} |\overline{K}(x, y; t_1) \overline{F}|_t &= e^{\mathcal{O}(t_1^{\epsilon/n})} |\overline{F}|_t [H(x, y; t_1) + \mathcal{O}(t_1)] \\ |\overline{F} \overline{K}(x, y; t_1)|_t &= e^{\mathcal{O}(t_1^{\epsilon/n})} |\overline{F}|_t [H(x, y; t_1) + \mathcal{O}(t_1)] \end{aligned} \quad (4.3)$$

and if $f(x, \psi_x) \in \Gamma(\Lambda^* TM)$.

$$\lim_{t \rightarrow 0} K(t) * f = f. \quad (4.4)$$

Proof: For Eq. (4.3), notice that \overline{K} can be written as $(P + \overline{L}) H$, where \overline{L} is a sum of products of terms which are either rank 2 and degree at most 1 (namely,

$e^{\frac{1}{12}\text{Ricci}_{kl}x^kx^l+\frac{t_1t_2}{12}}-1$, or $-\frac{1}{6}R_{kmjl}x^mx^j\psi^k\frac{\partial}{\partial\psi^l}$) or a multiple of t_1 and degree at most 2 (namely, $-\frac{t}{4}\left(\frac{\partial}{\partial\psi},\overline{R}[\psi,\psi]\frac{\partial}{\partial\psi}\right)$). The result follows from Eq. (2.5).

For Eq. (4.4) let R be the set of x for which $K(x,y;t)$ is nonzero for a given y . Notice $K(x,y,\psi_x,\psi_y;t)=He^{i\langle\rho,\psi_x-\psi_y\rangle}[1+\mathcal{O}(|\mathbf{y}_x|+t)]$, and, if f is smooth, $f(y,\psi_y)=f(x,\psi_y)+\mathcal{O}(|\mathbf{y}_x|)$, so

$$\begin{aligned}[K(t)*f](x,\psi_x) &= \int_R \oint K(x,y,\psi_x,\psi_y;t)f(y,\psi_y)d\psi_y dy \\ &= \int_R \oint \oint (2\pi t)^{-n/2}e^{-\frac{|\mathbf{y}_x|^2}{2t}+i\langle\rho,\psi_x-\psi_y\rangle}[f(x,\psi_y)+\mathcal{O}(|\mathbf{y}_x|+t)]d\rho d\psi_y d\mathbf{y}_x \\ &= \int_R (2\pi t)^{-n/2}e^{-\frac{|\mathbf{y}_x|^2}{2t}}[f(x,\psi_x)+\mathcal{O}(|\mathbf{y}_x|+t)]d\mathbf{y}_x \\ &= f(x,\psi_x)+\mathcal{O}(t^{1/2}).\end{aligned}$$

□

4.2 THE NORM ON KERNELS

Error terms arising in the following sections will typically be kernels of the form

$$[F_1(x,y;t)H(x,y;t)+F_2(x,y;t)]P(x,y)$$

where $F_1, F_2 \in E_{x,y}$. The size of $|F_i|_t$ gives a rough measure of the size of these errors, but more precision requires additional details on the size of F_2 .

Proposition 4.1 *Given ϵ as in §2.3, there exists a $D > 0$ such that, defining $\|L\|_t$ for $L \in E_{x,y}$ to be the least number such that*

$$|\overline{L}|_t \leq \|L\|_t[H(t)+Dt], \quad (4.5)$$

this norm satisfies, for K defined in Eq. (3.1) and for bounded and positive t_1 and t_2 with $t_1+t_2=t$,

$$\begin{aligned}\|K(t_1)*L\|_t &\leq e^{\mathcal{O}(t_1^{\epsilon/n})}\|L\|_{t_2} \\ \|L*K(t_2)\|_t &\leq e^{\mathcal{O}(t_2^{\epsilon/n})}\|L\|_{t_1}\end{aligned} \quad (4.6)$$

and

$$\|L_1*L_2\|_t \leq \mathcal{O}(t^{-1})\|L_1\|_{t_1}\|L_2\|_{t_2} \quad (4.7)$$

Proof: Observe that

$$\overline{L_1*L_2}(x,z) = \int \overline{L_1}(x,y)\mathfrak{P}_y^x\overline{L_2}(y,z)\mathfrak{P}_x^y dy,$$

and that $|\cdot|_t$ is unchanged by parallel transport. For the first line of Eq. (4.6), apply this in the special case $L_1 = K(t_1)$. Then, using Eq. (4.3) to define

$b_1, D > 0$, applying property (c) of Lemma 2.1 with $t_2 < t$, using Eq. (4.2) to define $b_2, D_2 > 0$, and using Eq. (4.1) to define D_3 ,

$$\begin{aligned}
\left| \overline{K(t_1) * L} \right|_t &\leq \int \left| \overline{K}(x, y; t_1) \mathfrak{P}_y^x \overline{L}(y, z) \mathfrak{P}_x^y \right|_t dy \\
&\leq e^{b_1 t_1^{\epsilon/n}} \int \left| \overline{L} \right|_t [H(x, y; t_1) + Dt_1] dy \\
&\leq e^{b_1 t_1^{\epsilon/n}} \|L\|_{t_2} \int [H(y, z; t_2) + Dt_2] [H(x, y; t_1) + Dt_1] dy \\
&\leq e^{b_1 t_1^{\epsilon/n}} \|L\|_{t_2} \left[e^{b_2 t_1^{\epsilon/n}} H(x, z; t) + Dt_2 + Dt_1 + D_2 t^2 \right. \\
&\quad \left. + DD_3 t_1 t_2 + DD_3 t_1 t_2 + D^2 t_1 t_2 \text{Vol}(M) \right] \\
&\leq e^{(b_1 + b_2) t_1^{\epsilon/n}} \|L\|_{t_2} H(x, z; t) + e^{b_1 t_1^{\epsilon/n}} \|L\|_{t_2} Dt \\
&\quad + e^{b_1 t_1^{\epsilon/n}} \|L\|_{t_2} Dt \left[D_3(t_1 + t_2) + \frac{D_2}{D} t + D \text{Vol}(M) t \right] \\
&\leq e^{b t_1^{\epsilon/n}} \|L\|_{t_2} [H(x, z; t) + Dt].
\end{aligned}$$

Here the D defining the norm in Eq. (4.5) is the constant coming from Eq (4.3), and b must satisfy $b > b_1 + b_2$ and $b t_1^{\epsilon/n} > b_1 t_1^{\epsilon/n} + D_3 t + D \text{Vol}(M) t + D_2 t/D$. Note such a b is guaranteed to exist since $t_1^{-\epsilon/n}$ times the right-hand side of this inequality is bounded, independent of t . The last line then follows from obvious estimates on $e^{(b-b_1) t_1^{\epsilon/n}}$, using $t = t_1 + t_2$ once more. The second version of Eq. (4.6) works in exactly the same fashion.

For Eq. (4.7), for some $C_1 > 0$ coming from Eq. (2.6), $C_2 > 0$ coming from Eq. (4.2) and $C_3 > 0$ coming from Eq. (4.1) and (4.2) (and depending on D)

$$\begin{aligned}
\left| \overline{L_1 * L_2} \right|_t &\leq \int \left| \overline{L_1}(x, y) \mathfrak{P}_y^x \overline{L_2}(y, z) \mathfrak{P}_x^y \right|_t dy \\
&\leq C_1 t^{-1} \int \left| \overline{L_1}(x, y) \right|_{t_1} \left| \overline{L_2}(y, z) \right|_{t_2} dy \\
&\leq C_1 \|L_1\|_{t_1} \|L_2\|_{t_2} t^{-1} \int [H(x, y; t_1) + Dt_1] [H(y, z; t_2) + Dt_2] dy \\
&\leq C_1 \|L_1\|_{t_1} \|L_2\|_{t_2} t^{-1} [C_2 H(x, z; t) + C_3 Dt] \\
&\leq C_4 \|L_1\|_{t_1} \|L_2\|_{t_2} t^{-1} [H(x, z; t) + Dt]
\end{aligned}$$

for $C_4 > C_1 \max(C_2, C_3)$. □

With this definition, Eq. 3.5 of Proposition 3.1 immediately implies

Corollary 4.1 *If ϵ is as in §2.3, K as in Eq. (3.1) and Δ as in Eq. (3.4)*

$$\begin{aligned}
\frac{\partial}{\partial t} K &= -\frac{1}{2} \Delta_x K + \mathcal{O}_t(t^\epsilon) \\
&= -\frac{1}{2} \Delta_y K + \mathcal{O}_t(t^\epsilon)
\end{aligned} \tag{4.8}$$

where $\mathcal{O}_t[f(t)]$ refers to an arbitrary kernel bounded in $\|\cdot\|_t$ by a multiple of $f(t)$.

4.3 APPROXIMATE SEMIGROUP PROPERTY

Proposition 4.2 For $K(t) = K(x, y, \psi_x, \psi_y; t)$ in Eq. (3.1), the norm $\|\cdot\|_t$ as in Proposition 4.1 and $t = t_1 + t_2 > 0$

$$\|K(t_1) * K(t_2) - K(t)\|_t = \mathcal{O}(t^{1+\epsilon}) \quad (4.9)$$

Proof:

Let r be the injectivity radius of M , so that $K(x, y; t) = 0$ if x and y are more than r apart. Suppose first that the spatial variables x and z implicit on the left-hand side of Eq. (4.9) are more than $r/2$ apart. By the triangle inequality, $|\overline{K(t_1) * K(t_2)} - \overline{K(t)}|_t \leq |\overline{K(t_1) * K(t_2)}|_t + |\overline{K(t)}|_t$. The second summand is bounded by a constant times $H(x, z; t)$, which, because the distance between x and z is bounded below, is exponentially damped for small t ; in particular, it is $\mathcal{O}(t^{1+\epsilon})$. The first summand is bounded by some non-negative power of t times $H(t_1) * H(t_2)$. Since within the integral represented by the $*$ product every point y in the domain is a distance at least $r/4$ from either x or z , this integral is likewise exponentially damped, which proves Eq. (4.9) in this case.

If x and z are less than $r/2$ apart,

$$\begin{aligned} K(t_1) * K(t_2) - K(t) &= \int_0^{t_1} \partial[K(\tau) * K(t - \tau)] / \partial \tau \, d\tau \\ &= \int_0^{t_1} \dot{K}(\tau) * K(t - \tau) - K(\tau) * \dot{K}(t - \tau) \, d\tau \\ &= \int_0^{t_1} \mathcal{O}_\tau(\tau^\epsilon) * K(t - \tau) + K(\tau) * \mathcal{O}_{t-\tau}([t - \tau]^\epsilon) \, d\tau \\ &\quad + \int_0^{t_1} -\Delta_y[K(\tau)] * K(t - \tau) \, d\tau + K(\tau) * \Delta_y[K(t - \tau)] \, d\tau \\ &= \int_0^{t_1} \mathcal{O}_t(\tau^\epsilon) + \mathcal{O}_t([t - \tau]^\epsilon) \, d\tau \\ &\quad + \int_0^{t_1} \left[\int_{\partial_y} \star d_y[K(\tau)] K(t - \tau) \, dy - K(\tau) \star d_y[K(t - \tau)] \, dy \right] \, d\tau \end{aligned}$$

where the third equation comes from Eq. (4.8), the fourth from Eq. (4.6), and boundary term from taking the formal adjoint of Δ_{LdR} . The \int_{∂_y} indicates the integral in the y variable of the given forms along the boundary of the earlier region of integration, and \star denotes the Hodge star operator.

The first integral reduces to $\mathcal{O}_t(t^{1+\epsilon})$. For the boundary integral, the domain of integration has bounded metric volume (bounded by the sum of the volumes of the spheres of the injectivity radius around x and z), so to bound the integral it suffices to bound the integrand. This integrand is of the form

a polynomial in $\tau, t - \tau, \tau^{-1}$, and $(t - \tau)^{-1}$ times a smooth function of x, y, z times $e^{-|y_x|^2/2\tau - |y_z|^2/2(t-\tau)}$. Each point on the boundary is a distance r from either x or z , and since x and z are less than $r/2$ apart, each point is a distance at least $r/2$ from both. Thus the Gaussian is exponentially damped in τ and $t - \tau$, so the whole expression is bounded by a multiple of any power of t_1 and t_2 . In particular it is $\mathcal{O}(t^{1+\epsilon}) = \mathcal{O}_t(t^\epsilon)$. Eq. (4.9) then follows upon completing the now trivial τ integral. \square

5 Partitions

5.1 PARTITIONS AND THE POINTWISE LIMIT

For $t > 0$, a partition P of t is a sequence $P = (t_1, t_2, \dots, t_m)$ with $t = \sum_i t_i$. If P is a partition of t and P' is a partition of t' , then the concatenation PP' is a partition of $t + t'$; if P_i is a partition of t_i for $1 \leq i \leq m$, then $P_1 P_2 \dots P_m$ is a refinement of $P = (t_1, \dots, t_m)$. Define $|P| = \max t_i$. If P is a partition of t , and K is any kernel, define

$$K^{*P} = K(t_1) * K(t_2) * \dots * K(t_m) \quad (5.1)$$

Take $\{P : |P| < r\}$ for $r > 0$ to define open sets in the space of all partitions of a given t . Then it makes sense to define $\lim_{|P| \rightarrow 0}$ of a function depending on t and P .

Proposition 5.1 *For K as defined in Eq. 3.1, $f_0(x, \psi_x)$ piecewise continuous, and $t \geq 0$,*

$$f(t) = \lim_{|P| \rightarrow 0} K^{*P} * f_0 \quad (5.2)$$

is the unique solution $f(x, \psi_x; t)$ to the heat equation $\partial f / \partial t = -\frac{1}{2} \Delta_{\text{LdR}} f$ with $f(0) = f_0$.

Proof: Let $f(t)$ be the solution to the heat equation with $f(0) = f_0$ (the dependence on x and ψ_x is understood). The definition of K ensures that $K(t) * f(s)$ is a smooth function of t with $\lim_{t \rightarrow 0} K(t) * f(s) = f(s)$. Then $f(t_1 + t_2) - f(t_2) = t_1 \Delta_{\text{LdR}} f(t_2) + \mathcal{O}(t_1^2)$ (all explicit and implicit constants in this proof depend on f and thus f_0). Using Eq. 4.4 to expand about $t_1 = 0$, and (4.8)

$$K(t_1) * f(t_2) - f(t_2) = t_1 \left. \frac{\partial K}{\partial t} \right|_{t=0} * f(t_2) + \mathcal{O}(t_1^2) = t_1 \Delta_{\text{LdR}} f(t_2) + \mathcal{O}(t_1^{1+\epsilon}).$$

This uses the fact that for $E(t) = \mathcal{O}_t(t^\epsilon)$, $E(t) * f = \mathcal{O}_t(t^\epsilon)$.

Thus there is a constant C such that $\|K(t_1) * f(t_2) - f(t_1 + t_2)\|_\infty \leq C t_1^{1+\epsilon}$ for all $t_1 + t_2 \leq t$. Since the operator $K*$ depends smoothly on t and approaches

the identity as t goes to 0 we can also select C so that the operator norm of $K(t_1)$ is less than e^{Ct_1} .

Therefore decomposing P for each $t_k \in P$ as $P_1(t_k)P_2$, and writing $[K^{*P} * f(0) - f(t)]$ as an appropriate telescoping sum,

$$\begin{aligned} \left\| [K]^{*P} * f(0) - f(t) \right\|_\infty &\leq \sum_k \left\| [K]^{*P_1} * \left[K(t_k) * f\left(\sum_{i>k} t_i\right) - f\left(t_k + \sum_{i>k} t_i\right) \right] \right\|_\infty \\ &\leq \sum_k e^{C \sum_{i<k} t_i} C t_k^{1+\epsilon} \leq C e^{Ct} |P|^\epsilon \end{aligned}$$

which converges to 0 under refinement. \square

5.2 THE CAUCHY PROPERTY

Proposition 5.2 *For sufficiently small $t > 0$, all partitions P of t , all refinements Q of P , and with the norm defined in Prop. 4.1*

$$\|K^{*Q} - K^{*P}\|_t = \mathcal{O}(t) |P|^\epsilon. \quad (5.3)$$

Proof: Eqs. (4.6), (4.7), and (4.9) give constants a, b, c such that for all $t = t_1 + t_2$ positive and bounded and all $L, L_1, L_2 \in E_{x,y}$

$$\begin{aligned} \|K(t_1) * K(t_2) - K(t)\|_t &\leq at^{1+\epsilon} \\ \|K(t_1) * L\|_t &\leq e^{bt_1^{\epsilon/n}} \|L\|_{t_2} \\ \|L * K(t_2)\|_t &\leq e^{bt_2^{\epsilon/n}} \|L\|_{t_1} \\ \|L_1 * L_2\|_t &\leq ct^{-1} \|L_1\|_{t_1} \|L_2\|_{t_2}. \end{aligned} \quad (5.4)$$

The proof breaks down into three lemmas which respectively provide positive constants a_1, b_1 , and b_2 such that for sufficiently small t (depending on a_i, b_i), and for every partition P of t

$$\|K^{*P} - K(t)\|_t \leq a_1 t^{1+\epsilon}, \quad (5.5)$$

for every $t_1 + t_2 = t$, every pair of partitions P_1 of t_1 and P_2 of t_2 , and every $L \in E_{x,y}$

$$\begin{aligned} \|K^{*P_1} * L\|_t &\leq e^{b_1 t_1^{\epsilon/n}} \|L\|_{t_2} \\ \|L * K^{*P_2}\|_t &\leq e^{b_1 t_2^{\epsilon/n}} \|L\|_{t_1}, \end{aligned} \quad (5.6)$$

and for all partitions P of t and all refinements Q of P ,

$$\|K^{*Q} - K^{*P}\|_t \leq a_1 e^{b_2 t^{\epsilon/n}} t |P|^\epsilon. \quad (5.7)$$

Since t is bounded this implies Eq. (5.3). \square

The proofs of Eqs. 5.5 and 5.7, provided in the lemmas below, rely on the following induction process: For any partition P of t , there exists a t_k in P such that, writing $P = P_1(t_k)P_2$, with P_i a partition of τ_i and $t = \tau_1 + t_k + \tau_2$, $\tau_i \leq t/2$. (the largest k so that $\tau_1 < t/2$.) Assuming a given property holds for P_1 and P_2 and proving it therefore holds for P will prove it holds in general, provided it also holds for the empty partition of 0 and the trivial partition (t) .

Lemma 5.1 *Eq. (5.5) holds.*

Proof: Begin with the special case $P = (t_1, t_2, t_3)$. Several applications of Eq. (5.4) imply there is an a_2 depending on a and b such that

$$\|K^{*P} - K(t)\|_t \leq a_2 t^{1+\epsilon} \quad (5.8)$$

in this case.

To begin the inductive argument, note Eq. (5.5) clearly holds when $t = 0$ or $P = (t)$. For the inductive step, write $P = P_1(t_k)P_2$, where P_i is a partition of $\tau_i \leq t/2$. Using the decomposition

$$|ABC - A'BC'| \leq |(A - A')BC'| + |A'B(C - C')| + |(A - A')B(C - C')|,$$

$$\begin{aligned} & \|K^{*P} - K(t)\|_t \\ &= \|K(\tau_1) * K(t_k) * K(\tau_2) - K(t) + K^{*P_1} * K(t_k) * K^{*P_2} - K(\tau_1) * K(t_k) * K(\tau_2)\|_t \\ &\leq \|K(\tau_1) * K(t_k) * K(\tau_2) - K(t)\|_t + a_1 \tau_1^{1+\epsilon} e^{bt_k^{\epsilon/n} + b\tau_2^{\epsilon/n}} \\ &\quad + a_1 e^{b\tau_1^{\epsilon/n} + bt_k^{\epsilon/n}} \tau_2^{1+\epsilon} + ct^{-1} a_1^2 e^{bt_k^{\epsilon/n}} \tau_1^{1+\epsilon} \tau_2^{1+\epsilon} \\ &\leq a_2 t^{1+\epsilon} + a_1 e^{2bt^{\epsilon/n}} 2^{-\epsilon} t^\epsilon \tau_1 + a_1 e^{2bt^{\epsilon/n}} 2^{-\epsilon} t^\epsilon \tau_2 + ca_1^2 e^{bt^{\epsilon/n}} 2^{-2-2\epsilon} t^{1+2\epsilon} \\ &\leq a_2 t^{1+\epsilon} + a_1 2^{-\epsilon} e^{2bt^{\epsilon/n}} t^{1+\epsilon} [1 + ca_1 2^{-2-2\epsilon} t^\epsilon]. \end{aligned}$$

The second inequality uses $t_k < t$ and $\tau_i < t/2$, the third is just factoring plus some easy estimates. For this to be less than $a_1 t^{1+\epsilon}$ requires that $f(a_1, t)$ be greater than a_2 , where $f(a_1, t) = a_1 - a_1 2^{-\epsilon} e^{2bt^{\epsilon/n}} [1 + ca_1 2^{-2-2\epsilon} t^\epsilon]$. When $t = 0$ this is true for sufficiently large a_1 , so by continuity there is a value of a_1 for which it is true for all t less than some bound. \square

Lemma 5.2 *Eq. (5.6) holds.*

Proof: Using Eqs. (5.5) and (5.4)

$$\begin{aligned}\|K^{*P_1} * L\|_t &\leq \|K(t_1) * L\|_t + ct^{-1}a_1t_1^{1+\epsilon}\|L\|_{t_2} \\ &\leq \left[e^{bt_1^{\epsilon/n}} + ca_1t_1^\epsilon\right]\|L\|_{t_2} \leq e^{b_1t_1^{\epsilon/n}}\|L\|_{t_2}.\end{aligned}$$

In the second inequality use Eq. (5.4) and bound t_1 by t , and in the third choose b_1 large enough that $e^{b_1t_1^{\epsilon/n}}$ bounds $\left[e^{bt_1^{\epsilon/n}} + ca_1t_1^\epsilon\right]$. \square

Lemma 5.3 *Eq. (5.7) holds.*

Proof: Write $Q = Q_1Q_2 \cdots Q_m$ where Q_i is a partition of t_i in P . Again proceed by induction on P , the empty case being trivial and the $P = (t)$ case following from Eq. (5.5). For the induction step, write $P = P_1(t_k)P_2$ with P_i a partition of $\tau_i \leq t/2$, and this time apply the decomposition

$$\begin{aligned}|ABC - A'B'C'| &= |(A - A')BC'| + |A'B(C - C')| + |A'(B - B')C'| \\ &\quad + |(A - A')B(C - C')|\end{aligned}$$

to get

$$\begin{aligned}\|K^{*Q} - K^{*P}\|_t &= \|K^{*Q_1} \cdots K^{*Q_{k-1}} * K^{*Q_k} * K^{*Q_{k+1}} \cdots K^{*Q_n} - K^{P_1} * K(t_k) * K^{*P_2}\|_t \\ &\leq e^{b_1t_k^{\epsilon/n} + b_1\tau_2^{\epsilon/n} + b_2\tau_1^{\epsilon/n}} a_1\tau_1 |P_1|^\epsilon + e^{b_1\tau_1^{\epsilon/n} + b_1t_k^{\epsilon/n} + b_2\tau_2^{\epsilon/n}} a_1\tau_2 |P_2|^\epsilon \\ &\quad + e^{b_1\tau_1^{\epsilon/n} + b_1\tau_2^{\epsilon/n}} a_1t_k^{1+\epsilon} + ca_1^2t^{-1}e^{b_1t_k^{\epsilon/n} + b_2\tau_1^{\epsilon/n} + b_2\tau_2^{\epsilon/n}} \tau_1\tau_2 |P_1|^\epsilon |P_2|^\epsilon \\ &\leq a_1e^{2b_1t^{\epsilon/n} + 2^{-\epsilon/n}b_2t^{\epsilon/n}} |P|^\epsilon \left[\tau_1 + \tau_2 + t_k + \frac{ca_1}{4}e^{b_2t^{\epsilon/n}}t |P|^\epsilon\right] \\ &\leq a_1e^{2^{-\epsilon/(2n)}b_2t^{\epsilon/n}}t |P|^\epsilon \left[1 + \frac{ca_1}{4}e^{b_2t^{\epsilon/n}} |P|^\epsilon\right].\end{aligned}$$

The first inequality follows from the inductive hypothesis Eq. (5.7), Eqs. (5.4), and Eq. (5.5). The second follows from $\tau_i < t/2$ and $t_k < t$ as before, along with $t_k < |P|$, $|P_i| < |P|$, and $\tau_1\tau_2 \leq t^2/4$. We can clearly choose b_2 big enough to get the third inequality. This quantity divided by $t |P|^\epsilon$ is clearly less than $a_1e^{b_2t^{\epsilon/n}}$ when $t = 0$, so it must be for sufficiently small t ($|P|$ is bounded by t of course). \square

6 The Large Partition Limit

Theorem 6.1 *For sufficiently small, positive t the pointwise limit*

$$K^\infty(t) = \lim_{|P| \rightarrow 0} K^{*P} \tag{6.1}$$

exists in $E_{x,y}$ and is smooth in t . In fact, for all partitions and any $0 < \epsilon < 1/2$

$$\|K^\infty(t) - K^{*P}\|_t = \mathcal{O}(t|P|^\epsilon), \quad (6.2)$$

and, in particular,

$$\|K^\infty(t) - K(t)\|_t = \mathcal{O}(t^{1+\epsilon}). \quad (6.3)$$

Moreover, the limit $K^\infty(t)$ is the heat kernel for the Laplace-de Rham operator, satisfying

$$\frac{\partial}{\partial t} K^\infty = -\frac{1}{2} \Delta_{LdR} K^\infty \quad (6.4)$$

and for every form $f \in \Gamma(\Lambda^* TM)$

$$\lim_{t \rightarrow 0} K^\infty(t) * f = f. \quad (6.5)$$

Proof: For a fixed t , The expression $H(t) + Dt$ used to define the t -norm in Prop. 4.1 is bounded above and below, so the t -norm $\|\cdot\|_t$ is bounded above and below by a multiple of the supremum norm. Therefore, fixing some sequence P_n of refinements with $\lim_{n \rightarrow \infty} |P_n| = 0$ the sequence K^{*P_n} is Cauchy by Eq. (5.3) and thus converges to some $K^\infty(t)$ continuous in t and the suppressed x and y . In particular

$$\|K^\infty(t) - K^{*P_n}\|_t = \mathcal{O}(t|P_n|^\epsilon).$$

If P is any other partition of t then for each P_n with $|P_n| \leq |P|$ there is a P' which is a common refinement of P and P_n , and therefore by Eq. (5.3) again all the quantities $K^{*P'}$, K^{*P} , K^{*P_n} , and $K^\infty(t)$ differ by $\mathcal{O}(t|P|^\epsilon$, proving Eqs. (6.1), (6.2), and (6.3).

If $f(x, \psi_x)$ is any smooth form on M , there exists for small enough $t > 0$ a smooth form $f(x, \psi_x; t)$ satisfying the heat equation

$$\frac{\partial}{\partial t} f = -\frac{1}{2} \Delta_{LdR} f$$

and

$$f(x, \psi_x; 0) = f(x, \psi_x).$$

By Eqs. (5.2) and (6.2)

$$K^\infty(t) * f = f(t)$$

proving Eq. (6.5). Thus as a distribution K^∞ is a solution to Eq. (6.4), and therefore, since the Laplace-de Rham heat equation is elliptic, by elliptic regularity ([Eva98]) it is in fact smooth in x , y , and t and satisfies Eq. (6.4). \square

Corollary 6.1 *In fact K^∞ is defined for all positive t by*

$$K^\infty(t) = \lim_{|P| \rightarrow 0} [K(t_i)]^{*P}, \quad (6.6)$$

and is the heat kernel satisfying Eqs. (6.4) and (6.5). The above limit converges pointwise and, for fixed t , uniformly.

Proof: Suppose Theorem 6.1 applies for all positive t less than some $T > 0$. For such t , K^∞ satisfies the semigroup property $K^\infty(t_1) * K^\infty(t_2) = K^\infty(t)$ for $t = t_1 + t_2$, since the two kernels give the same operator on forms. Extending the definition of $K^\infty(t)$ for all positive t to agree with $(K^\infty)^{*P}$ for P any partition of t fine enough that $|P| < T$ is therefore well-defined. This extended $K^\infty(t)$ still satisfies Eqs. (6.4) and (6.5).

To see that this extended $K^\infty(t)$ satisfies Eq. (6.6), consider any partition Q with $|Q| < T/2$, so that it can be written as a refinement of a partition P where each t_i satisfies $T/2 \leq t_i < T$. That is, $Q = Q_1 \cdots Q_m$, where Q_i is a partition of t_i . Eq. (6.2) and $|Q_i| \leq |Q|$, guarantee

$$\|K^{*Q_i} - K^\infty(t_i)\|_{t_i} = \mathcal{O}(t_i) |Q|^\epsilon.$$

Since t_i is bounded above and below $\|\cdot\|_{t_i}$ is commensurate with the sup norm, so there is a $b > 0$ such that

$$\|K^{*Q_i} - K^\infty(t_i)\|_\infty \leq bt_i |Q|^\epsilon.$$

Also, because K^∞ is the kernel of an elliptic operator there is a $c > 0$ such that $\|K^\infty(t) * f\|_\infty \leq e^{ct} \|f\|_\infty$. Induction on m will prove $\|K^{*Q} - K^\infty(t)\|_\infty \leq e^{at} |Q|^\epsilon$. For the induction step, apply the decomposition

$$|AB - A'B'| \leq |(A - A')B'| + |A'(B - B')| + |(A - A')(B - B')|$$

to get

$$\begin{aligned} \|K^{*Q} - K^\infty(t)\|_\infty &= \|K^{*Q_1 \cdots Q_{m-1}} * K^{*Q_m} - K^\infty(t - t_m) * K^\infty(t_m)\|_\infty \\ &\leq \left[e^{a(t-t_m)+ct_m} + e^{c(t-t_m)} bt_m + \text{Vol}(M) e^{a(t-t_m)} bt_m |Q|^\epsilon \right] |Q|^\epsilon \\ &\leq e^{a(t-t_m)} [e^{ct_m} + b_1 t_m] |Q|^\epsilon \\ &\leq e^{at} |Q|^\epsilon \end{aligned}$$

assuming in the second inequality that $a \geq c$ and choosing $b_1 \geq b + \text{Vol}(M) b T^\epsilon$ and in the third assuming that $a \geq c + b_1$. Eq. (6.6) follows. \square

Remark 6.1 In [FS08] the authors prove a weaker result for a kernel K^{mq} similar to K . There the result was that the limit taken along some sequence of refinements of P exists and agrees pointwise with the heat kernel. The stronger result depends on modifying the approximating kernel to agree with the heat kernel (as a distribution) to higher order in t . Both approximate kernels are consistent with time-slicing the imaginary-time path integral, as they amount to different interpolations between the fixed end-values.

Remark 6.2 *The choice of K as the approximate kernel, and the basis for a discrete approximation to the path integral, is, as noted in the introduction, subject to some ambiguity in how the path integral specifies a discrete approximation. It is natural to ask how the results of Theorem 6.1 and Corollary 6.1 depend on the exact choice of K . Most straightforwardly, adding terms of the form $\mathcal{O}(t^2)H$ or $\mathcal{O}(|\mathbf{y}_x|^3)H$, and using the standard norm in place of $|\cdot|_t$, it is easy to follow along with the argument and see it all goes through unchanged. This would not be sufficient to prove the Gauss-Bonnet-Chern Theorem, but in most other respects seems as powerful a result. More subtly, adding to K a term of the form $H[\langle \mathbf{y}_x, \mathbf{A} \mathbf{y}_x \rangle - \text{tr} A]$, where A is section of $\text{End}(T_x M)$, does not change the limit, although the above argument in itself would not suffice and the convergence is slower. In particular Eq. (6.3) would no longer hold (even with a modified norm). Thus while $K = He^{t\tau/6}$, which would appear to be the simplest choice, will converge in the large partition limit to the heat kernel for the Laplace-Beltrami operator, only $K = He^{t\tau/12 + t\text{Ric}(\mathbf{y}_x, \mathbf{y}_x)/12}$ offers an analogue of Eq. (3.5) and hence Eqs. (6.2) and (6.3).*

7 The Gauss-Bonnet-Chern Theorem

Theorem 7.1

$$\left[\lim_{t \rightarrow 0} K^\infty(x, x, \psi_x, \psi_x; t) \right]_{\text{top}} = (2\pi)^{-n/2} \text{Pfaff}(R) \quad (7.1)$$

Remark 7.1 *McKean and Singer [MS67] prove the integral of the matrix supertrace of the heat kernel, i.e. $\int \oint K^\infty(x, x, \psi_x, \psi_x; t) d\psi_x dx$, is equal to the signed sum of the Betti numbers, independent of t . Thus, the theorem implies the Gauss-Bonnet-Chern theorem, thereby completing a rigorous version of the path-integral proof of the latter. In fact, it implies a local version of the Gauss-Bonnet-Chern theorem:*

$$\lim_{t \rightarrow 0} \text{str} K^\infty(x, x; t) = (2\pi)^{-n/2} \text{Pfaff}(R)(x).$$

Proof: Theorem 6.1, in particular Eq. (6.3), estimates the heat kernel K^∞ as the approximation K plus an error term whose t -norm is $\mathcal{O}(t^{1+\epsilon})$. That is, $K^\infty(t) - K(t) = F_1 H + F_2 t$, where $|F_i|_t = \mathcal{O}(t^{1+\epsilon})$, so the supertrace of the error term is bounded by

$$\text{str} F_1 (2\pi t)^{-n/2} + \text{str} F_2 t.$$

Lemma 2.2 implies $\text{str} F_i = \mathcal{O}(t^{n/2+2\epsilon/n})$ so the supertrace of the error term is bounded by $\mathcal{O}(t^{2\epsilon/n})$ and therefore goes to zero with t . On the other hand by

Eq. (3.1)

$$\begin{aligned}
\lim_{t \rightarrow 0} \oint K(x, x, \psi, \psi; t) &= \lim_{t \rightarrow 0} (2\pi t)^{-n/2} \oint \exp \left[-\frac{t\mathfrak{r}}{6} + \frac{t}{4}(\rho, R[\psi, \psi] \rho) \right] d\rho \\
&= (2\pi)^{-n/2} \oint e^{\frac{1}{4}(\rho, R[\psi, \psi] \rho)} d\rho = (2\pi)^{-n/2} \text{Pfaff} \left(\frac{1}{2} R[\psi, \psi] \right) \\
&= (2\pi)^{-n/2} \text{Pfaff}(R).
\end{aligned}$$

8 Some conclusions

The heuristic path integral argument leading to the Gauss-Bonnet-Chern and other index theorems has two key premises: first, the path integral for supersymmetric quantum mechanics, taken over free loops, gives the supertrace of the Laplace-de Rahm heat kernel; second, steepest descent applies to the path integral to give the Pfaffian of the curvature as its small- t limit. Proposition 3.1 and Eq. 6.3 of Theorem 6.1 give the precise sense in which the kernel K approximates the heat kernel for small t . The product K^{*P} thus represents the approximate imaginary-time path integral for paths with fixed endpoints, in accord with Feynman's time-slicing approach to combining a choice of partition P with the small- t approximation. Corollary 6.1 to Theorem 6.1 then proves, that, with this precise definition, the path integral with fixed endpoints agrees with the heat kernel. This immediately leads to the first premise of the heuristic argument. For the second, Eq. 6.3 proves the steepest-descent approximation applies to this definition of the path integral, and Theorem 7.1 shows this approximation agrees with the Pfaffian in the small- t limit.

In short, $\lim_{|P| \rightarrow 0} K^{*P}$ provides a rigorous definition for the path integral in which the path integral argument for GBC carries over directly. As such, it may serve as a template for rigorous definition path integrals for such immediate generalizations as $N = 1/2$ supersymmetric quantum mechanics, which the authors believe requires only minor changes to the definition of K and should give a path integral proof of the Atiyah-Singer index theorem. More generally, it is not unreasonable to hope this approach will generalize to a wide range of cohomological field theories, because each of these depend, at least heuristically, on the same localization to the steepest-descent (or stationary-phase) approximation.

Looked at another way, the heuristic arguments rely on an unstated assumption about the time-slicing prescription for the path integral; namely, the order of taking the small- t and small- $|P|$ limits does not matter. The application of Lemma 2.2 in the proof of Theorem 7.1 shows, in conjunction with the Cauchy property of Proposition 5.2, that, even for the path integral corresponding to based loops, the small- t limit of K^∞ agrees with the small- $|P|$ limit of $K^{*P}(0)$.

In their seminal work on the heat kernel McKean and Singer [MS67] referred to “fantastic cancellations” which must occur amongst lower-order terms in the asymptotic expansion for the heat kernel so that its supertrace agrees with that of the Pfaffian of the curvature, in keeping with the Gauss-Bonnet-Chern

theorem. In the present construction of the path integral, according to Eq. 6.3, the asymptotic expansion for the heat kernel begins with terms which on the diagonal are $\mathcal{O}(t^{-n/2})$, yet the supertrace is $\mathcal{O}(1)$. This happens because, first, the approximating kernel K already has this property, and second, the t -norm was devised so that the error terms, which are $\mathcal{O}_t(t^{1+\epsilon})$, have supertrace in $\mathcal{O}(t^{\epsilon/n})$.

□

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